SOLUTION OF LARGE, SPARSE SYSTEMS IN DESIGN AND ANALYSIS

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Abstract

Presents some basic principles of the solution of large systems of simultaneous linear equations by sparsity methods, particularly Zollenkopf's bifactorisation method, and describes diakoptics (Gabriel Kron's method of tearing). Implications of these two approaches, used together, are discussed with particular reference to system optimization and finite-element analysis of nonlinear and large problems.

Introduction

The manipulation and solution of sets of simultaneous equations is basic to most of the techniques employed in computer solution of field problems, network analysis and optimization methods.

In many problems the frequent attribute of sparseness, meaning that few nonzero elements exist, has permitted the design of special algorithms that allow practical problems to be solved. For example finite difference methods, that produce large sparse matrices of order of 5 000 - 20 000 [1,2], are conveniently solved by the use of the successive overrelaxation (SOR) iterative scheme with all unknowns kept in the fast store.

Direct methods, that treat matrices as dense, are necessarily limited to about 150 unknowns in a reasonably large modern computer. On the other hand direct methods employing sparsity techniques, and clever scheduling of disk transfers, allow solution of large systems. These solutions can be obtained without the inconvenience of having to guess at the acceleration factor and to suffer the possibility of erratic convergence behaviour and associated computational costs.

Sparsity methods have been used most notably in the solution of networks (e.g.electrical power systems [3,5] and structural analysis) and in linear programming optimization methods [6]. A good review of some microwave applications is given in [14]. Insofar as nonlinear optimization problems are concerned the Newton-Raphson approach has been used in load-flow problems and recently Sasson and others [7] have employed sparsity techniques in the gradient method applied to the same problem. There can be no doubt that general optimization methods (e.g. Bandler [8] and Charalambous [13]) will experience significant improvements when sparsity methods are employed for large systems.

Objectives

The objectives of this tutorial paper are as follows:

(a) To present an introduction to the principles behind the solution of large systems of equations through direct methods that exploit matrix sparsity. The introduction begins with a review of Gaussian elimination and back-substitution and its relationship to LU decomposition, symmetric Choleski factorization, and its application to band-storage and sparsity techniques. The importance of employing an optimal elimination sequence, in order to maintain sparsity, is pointed out.

(b) To present solution schemes for the repeated and efficient solution of $A\underline{x} = \underline{b}$ for many right-hand sides, in which case one would like to have the inverse of A available as a matter of economy. Since A^{-1} can be dense, although A is sparse, the storage of an explicit inverse is generally an impossible task for large problems. A class of efficient techniques is based upon the computation of very sparse matrix factors of A^{-1} which may be packed into approximately the same storage space as A itself. A very useful method, which is outlined, is due to Zollenkopf [9].

(c) To describe diakoptics (Gabriel Kron's method of tearing) with application to electrical networks [10,11] and finite element matrices [12]. This method holds great promise for the optimization of large systems in which it would be intolerable to invert the entire impedance matrix each time a small portion of the network is changed. An analogous situation in finite-element analysis is the solution of a field in a nonlinear medium (e.g. the saturation of a pole tip on the rotor of an electrical machine or the optimization of a portion of a microstrip structure). Again, as only a small portion of the square finite-element matrix is altered, it seems wasteful to invert the entire matrix many times. Diakoptics permits the nonlinear portion to be excised, the matrix corresponding to the linear portion to be inverted (through Zollenkopf's method, preferably), and the contribution due to the nonlinear portion to be blended into the larger system at a very small additional cost. In this way successive iteration, for the solution of nonlinear problems may be relatively cheaply done. In effect, it involves recomputation and inversion of the nonlinear portion alone. It should be pointed out that diakoptics supplies an exact solution of a dissected linear system and it is not a block-iterative scheme. Iteration is required for a nonlinear problem, however, due to changing parameters.

An immensely important function for diakoptics is in handling huge problems that cannot be held entirely in the fast store. In such cases it may be possible to tear the total problem into subsections, each of which can only just be accommodated in the fast store. Each subsystem can then be inverted in turn while subsequently transferring blocks of numbers to and from the backing store in sequence. In this way, huge systems can be solved economically. A byproduct of the diakoptical approach is that whereas the large system may be highly ill-conditioned, the individual subsystem will likely suffer much less from this form of instability. As a consequence, there is reason to expect that the final answers will be more accurate when performed in parts.

It is not intended that this paper survey the entire field. Rather the author hopes to indicate, on the basis of somewhat limited experience, what his bias is in terms of the solution of a large class of engineering problems - and to express his reasons for these choices.

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